# Variational approximation for two-time correlation functions in $\Phi^4$ theory : optimization of the dynamics

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**Abstract**

We apply the time-dependent variational principle of Balian and Vénéroni to the $\Phi^4$ theory. An appropriate parametrization for the variational objects allows us to write coupled dynamical equations from which we derive approximations for the two-time correlation functions involving two, three or four field operators.

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# 1 Introduction

A basic tool of quantum field theory is the generating functional for multi-time correlation functions. In this paper, we show how the time-dependent variational principle of Balian and Veneroni [1] can be implemented in the context of quantum field theory out of equilibrium. From this approach, we derive approximate non perturbative dynamical equations for expectation values and correlation functions of local and composite operators. These evolution equations could be useful to evaluate the fluctuations of the matter density in cosmological scenarios [2]. They could also provide informations about the non-equilibrium dynamics during the expansion of the quark-gluon plasma [3, 4]. We consider here the case of a self-interacting scalar field in a Minkowski metric described by the hamiltonian $H = \int d^d x \, \mathcal{H}(\vec{x})$ with

$$\mathcal{H}(\vec{x}) = \frac{1}{2}\,\Pi^2(\vec{x}) + \frac{1}{2}\,(\vec{\nabla}\Phi(\vec{x}))^2 + \frac{m_0^2}{2}\,\Phi^2(\vec{x}) + \frac{b}{24}\,\Phi^4(\vec{x}) \ . \tag{1.1}$$

We work in $d$ spatial dimensions. The constants $m_0$ and $b$ are respectively the bare mass and the bare coupling constant.

# 2 Variational evaluation of the generating functional

Let us define an operator $A(J,t)$, functional of the sources $J$ and function of the time $t$, having the form :

$$\begin{aligned} A(J,t) \ &= T \exp i \int_t^\infty dt' \ \Big\{ \int d^d x \ \ J^\Phi(\vec{x},t')\Phi^H(\vec{x},t',t) + J^\Pi(\vec{x},t')\Pi^H(\vec{x},t',t) \\ &+ \int d^d x \, d^d y \ \ J^{\Phi\Pi}(\vec{x},\vec{y},t') \ \Big(\Phi^H(\vec{x},t',t)\Pi^H(\vec{y},t',t) + \Pi^H(\vec{y},t',t)\Phi^H(\vec{x},t',t)\Big) \quad , \\ &+ J^{\Phi\Phi}(\vec{x},\vec{y},t') \ \Phi^H(\vec{x},t',t)\Phi^H(\vec{y},t',t) + J^{\Pi\Pi}(\vec{x},\vec{y},t') \ \Pi^H(\vec{x},t',t)\Pi^H(\vec{y},t',t)\Big\} \end{aligned} \tag{2.1}$$

where $T$ is the time-ordering operator and $\Phi^H(\vec{x},t',t)$ and $\Pi^H(\vec{x},t',t)$ are respectively the field and momentum operators defined in the Heisenberg representation. They satisfy the boundary conditions : $\Phi^H(\vec{x},t,t) = \Phi(\vec{x})$ and $\Pi^H(\vec{x},t,t) = \Pi(\vec{x})$, $\Phi(\vec{x})$ and $\Pi(\vec{x})$ being the operators in the Schrödinger representation. The local as well as bilocal sources $J(t')$

are turned on between $t' = t$ and $t' = +\infty$. The generating functional for the causal Green functions writes

$$Z(J, t_0) = Tr\left(D(t_0)\; A(J, t_0)\right) \;, \tag{2.2}$$

$D(t_0)$ being the initial state. The statistical operator $D(t_0)$ may represent a thermal equilibrium or a non-equilibrium situation. At zero temperature, it reduces to a projection operator. Besides the sources $J$, the generating functional $Z$ depends on the initial time $t_0$. We want to evaluate the functional derivatives of

$$W(J, t_0) = -i\; \ln Z(J, t_0) \;, \tag{2.3}$$

since $W(J, t_0)$ is the generating functional for the connected Green functions. Its expansion in powers of the sources writes :

$$\begin{array}{l} W(J, t_0) = -i\, n_0 + \int_{t_0}^{+\infty} dt' \,\{\int d^dx\; J_\Phi(\vec{x}, t')\; \varphi(\vec{x}, t') + \int d^dx\; d^dy\; J_{\Phi\Phi}(\vec{x}, \vec{y}, t')\; G(\vec{x}, \vec{y}, t') + ...\} \\ +\frac{i}{2} \int \int_{t_0}^{+\infty} dt'\; dt'' \,\{\int d^dx_1\; d^dx_2\; J_\Phi(\vec{x}_1, t')\; J_\Phi(\vec{x}_2, t'')\; C^2_{\Phi\Phi}(\vec{x}_1, \vec{x}_2, t', t'') + ...\} \\ +... \end{array} , \tag{2.4}$$

where $n_0 = TrD(t_0)$ ( we choose not to normalize $D(t_0)$). We use the following notations for the expectation values of one and two field operators :

$$\varphi(\vec{x}, t) = \frac{1}{n_0}\, Tr\left(\Phi^H(\vec{x}, t, t_0)\; D(t_0)\right) \;, \tag{2.5}$$

$$G(\vec{x}, \vec{y}, t) = \frac{1}{n_0}\, Tr\left(\Phi^H(\vec{x}, t, t_0)\; \Phi^H(\vec{y}, t, t_0)\; D(t_0)\right) - \varphi(\vec{x}, t)\; \varphi(\vec{y}, t) \;. \tag{2.6}$$

The two-time causal functions between two field operators are defined according to :

$$C^2_{\Phi\Phi}(\vec{x}_1, \vec{x}_2, t_1, t_2) = \frac{1}{n_0}\, Tr\left(T\; \Phi^H(\vec{x}_1, t_1, t_0)\; \Phi^H(\vec{x}_2, t_2, t_0)\; D(t_0)\right) - \varphi(\vec{x}_1, t_1)\; \varphi(\vec{x}_2, t_2) \;. \tag{2.7}$$

Similarly, we define $C^2_{\Phi\Pi}$, $C^2_{\Pi\Pi}$. We define also the three-point and four-point two-time correlation function $C^3$ and $C^4$ according to :

$$\begin{array}{l} \frac{1}{i} \quad \frac{\delta^2 W}{\delta J_\Phi(\vec{x}, t') \delta J_{\Phi\Phi}(\vec{y}, \vec{z}, t")}|_{J=0} \equiv C^3(\vec{x}, \vec{y}, \vec{z}, t', t") \\ \quad +\varphi(\vec{y}, t")\; \left(C^2(\vec{x}, \vec{z}, t', t") + \varphi(\vec{x}, t')\; \varphi(\vec{z}, t")\right) + \varphi(\vec{z}, t")\; \left(C^2(\vec{x}, \vec{y}, t', t") + \varphi(\vec{x}, t')\; \varphi(\vec{y}, t")\right) \;, \end{array} \tag{2.8}$$

$$\frac{1}{i}\quad \frac{\delta^2 W}{\delta J_{\Phi\Phi}(\vec{x},\vec{y},t')\delta J_{\Phi\Phi}(\vec{z},\vec{u},t")}|_{J=0} \equiv C^4(\vec{x},\vec{y},\vec{z},\vec{u},t',t")$$
$$+\Big(C^2(\vec{x},\vec{z},t',t") + \varphi(\vec{x},t')\,\varphi(\vec{z},t")\Big)\ \Big(C^2(\vec{y},\vec{u},t',t") + \varphi(\vec{y},t')\,\varphi(\vec{u},t")\Big) \tag{2.9}$$
$$+\Big(C^2(\vec{x},\vec{u},t',t") + \varphi(\vec{x},t')\,\varphi(\vec{u},t")\Big)\ \Big(C^2(\vec{y},\vec{z},t',t") + \varphi(\vec{y},t')\,\varphi(\vec{z},t")\Big)\ .$$

We will use the time-dependent variational principle of Balian and Veneroni [1] to obtain an approximation for our quantity of interest, $Tr(D(t_0)A(J,t_0))$. We define therefore the action functional

$$\mathcal{Z}\left(\mathcal{A}(t),\mathcal{D}(t)\right) = Tr\left(\mathcal{A}(t_0)\,\mathcal{D}(t_0)\right) + \mathcal{Z}_{dyn}\ , \tag{2.10}$$

with

$$\mathcal{Z}_{dyn} = Tr\int_{t_0}^{\infty} dt\ \mathcal{D}(t)\left\{\frac{d\mathcal{A}(t)}{dt} - i\left[\mathcal{A}(t),\int d^dx\mathcal{H}(\vec{x})\right] + i\mathcal{A}(t)\left(\sum_j J_j(t)Q_j\right)\right\}\ , \tag{2.11}$$

where we have written in a compact form the term which involves the sources $J$ :

$$\sum_j J_j(t)Q_j = \int d^dx\ \Big(J^{\Phi}(\vec{x},t)\Phi(\vec{x}) + J^{\Pi}(\vec{x},t)\Pi(\vec{x})\Big)$$
$$+\int d^dx\ d^dy\ J^{\Phi\Pi}(\vec{x},\vec{y},t)\left(\Phi(\vec{x})\Pi(\vec{y}) + \Pi(\vec{y})\Phi(\vec{x})\right) \tag{2.12}$$
$$+\int d^dx\ d^dy\ \Big(J^{\Phi\Phi}(\vec{x},\vec{y},t)\Phi(\vec{x})\Phi(\vec{y}) + J^{\Pi\Pi}(\vec{x},\vec{y},t)\Pi(\vec{x})\Pi(\vec{y})\Big)\ .$$

The variational quantities of the functional $\mathcal{Z}$, which depends on the sources $J$, are the observable-like and density-like operators $\mathcal{A}(t)$ and $\mathcal{D}(t)$. We look for the stationarity of $\mathcal{Z}$ with respect to variations of $\mathcal{A}(t)$ and $\mathcal{D}(t)$ subject to the boundary conditions : $\mathcal{A}(t=+\infty) = 1$ and $\mathcal{D}(t_0) = D(t_0)$, where $D(t_0)$ is the initial statistical operator which we assume to be given and equal to a Gaussian density matrix. The generating functional for the connected Green functions will be approximated by $W(J,t_0) = -i\ \ln\mathcal{Z}_{st}$.

We restrict ourselves to trial operators $\mathcal{A}(t)$ and $\mathcal{D}(t)$ which are exponentials of quadratic and linear forms of the field operators $\Phi(\vec{x})$ and $\Pi(\vec{x})$ (which we shall loosely call Gaussian operators). Let us introduce the operators $\mathcal{T}_b = \mathcal{A}(t)\,\mathcal{D}(t)$ and $\mathcal{T}_c = \mathcal{D}(t)\,\mathcal{A}(t)$ [5]. We define the following expectation values :

$$n(t) = Tr\mathcal{T}_b = Tr\mathcal{T}_c\ , \tag{2.13}$$

$$\varphi_b(\vec{x},t) = \frac{1}{n(t)}\,Tr(\mathcal{T}_b\Phi(\vec{x}))\ , \tag{2.14}$$

$$\pi_b(\vec{x},t) = \frac{1}{n(t)}\, Tr(\mathcal{T}_b \Pi(\vec{x}))\ , \tag{2.15}$$

$$G_b(\vec{x},\vec{y},t) = \frac{1}{n(t)}\, Tr(\mathcal{T}_b \tilde{\Phi}(\vec{x})\tilde{\Phi}(\vec{y}))\ , \tag{2.16}$$

$$S_b(\vec{x},\vec{y},t) = \frac{1}{n(t)}\, Tr(\mathcal{T}_b \tilde{\Pi}(\vec{x})\tilde{\Pi}(\vec{y}))\ , \tag{2.17}$$

$$T_b(\vec{x},\vec{y},t) = \frac{1}{n(t)}\, Tr(\mathcal{T}_b (\tilde{\Phi}(\vec{x})\tilde{\Pi}(\vec{y}) + \tilde{\Pi}(\vec{y})\tilde{\Phi}(\vec{x})))\ , \tag{2.18}$$

with $\tilde{\Phi}(\vec{x}) = \Phi(\vec{x}) - \varphi_b(\vec{x},t)$ and $\tilde{\Pi}(\vec{x}) = \Pi(\vec{x}) - \pi_b(\vec{x},t)$. Similarly we define the expectation values associated with $\mathcal{D}(t)\mathcal{A}(t)$ and indexed by $c$ by replacing $\mathcal{T}_b$ by $\mathcal{T}_c$ in (2.14)-(2.18). For $\mathcal{D}(t) = 1$, one has $\varphi_b(\vec{x},t) = \varphi_c(\vec{x},t) = \varphi_a(\vec{x},t)$ (and similar definitions for $\pi_a, G_a, S_a, T_a$). For $\mathcal{A}(t) = 1$, one has $\varphi_b(\vec{x},t) = \varphi_c(\vec{x},t) = \varphi_d(\vec{x},t)$ (and similar definitions for $\pi_d, G_d, S_d, T_d$). A Gaussian operator is completely characterized by a set $\{\varphi, \pi, G, S, T\}$. It will be convenient to introduce a vector $\alpha(\vec{x},t)$

$$\alpha(\vec{x},t) = \begin{pmatrix} \varphi(\vec{x},t) \\ -i\,\pi(\vec{x},t) \end{pmatrix}\ , \tag{2.19}$$

and a matrix $\Xi(\vec{x},\vec{y},t)$

$$\Xi(\vec{x},\vec{y},t) = \begin{pmatrix} 2\,G(\vec{x},\vec{y},t) & -i\,T(\vec{x},\vec{y},t) \\ -i\,T(\vec{y},\vec{x},t) & -2\,S(\vec{x},\vec{y},t) \end{pmatrix}\ . \tag{2.20}$$

We give in appendix the expressions of $\alpha_b, \alpha_c, \Xi_b$ and $\Xi_c$ as functions of the independent variational quantities $\alpha_a, \Xi_a$ and $\alpha_d, \Xi_d$ which characterize $\mathcal{A}$ and $\mathcal{D}$ respectively.

With this choice of the trial spaces for $\mathcal{A}(t)$ and $\mathcal{D}(t)$ , the Wick theorem allows us to express the functional $\mathcal{Z}$ in the form :

$$\begin{aligned} \mathcal{Z} = \quad & n(t_0) + \int_{t_0}^{+\infty} dt\ \Big[ \frac{dn}{dt}|_{\mathcal{D}(t)=cte} - i\,n(t) \int d^dx\ \left(\mathcal{E}_c(\vec{x},t) - \mathcal{E}_b(\vec{x},t)\right) \\ & + i\,n(t) \int d^dx_1 d^dx_2\, \mathcal{K}_c(\vec{x}_1,\vec{x}_2,t) \Big] \end{aligned}\ , \tag{2.21}$$

where the energy density $\mathcal{E}(\vec{x},t) = Tr\,(\mathcal{D}(t)\,\mathcal{H}(\vec{x},t))$ is given, for the Hamiltonian density (1.1), by :

$$\begin{aligned} \mathcal{E}(\vec{x},t) = \quad & \frac{1}{2}\,\pi^2(\vec{x},t) + \frac{1}{2}\,(\vec{\nabla}\varphi)^2 + \frac{m_0^2}{2}\,\varphi^2(\vec{x},t) + \frac{b}{24}\,\varphi^4(\vec{x},t) \\ & + \frac{1}{2}\,S(\vec{x},\vec{x},t) - \frac{1}{2}\,\Delta G(\vec{x},\vec{y},t)|_{x=y} + \frac{m_0^2}{2}\,G(\vec{x},\vec{x},t) \\ & + \frac{b}{8}\,G^2(\vec{x},\vec{x},t) + \frac{b}{4}\,\varphi^2(\vec{x},t)\,G(\vec{x},\vec{x},t)\ . \end{aligned} \tag{2.22}$$

The last term of the functional (2.21), $K_c = Tr\left(\mathcal{T}_c \sum_j J_j(t)\, Q_j\right) = \int d^dx_1 d^dx_2 \mathcal{K}_c(x_1, x_2, t)$, involves the sources $J$ and the expectation values indexed by $c$.

# 3 Dynamical equations

By varying the expression (2.21) with respect to $n_d, \alpha_d, \Xi_d$, with the boundary conditions

$$n_d(t_0) = n_0 \ , \ \ \alpha_d(t_0) = \alpha_0 \ , \ \ \Xi_d(t_0) = \Xi_0 \ , \tag{3.1}$$

where $n_0, \alpha_0$ et $\Xi_0$ characterize the initial Gaussian state $D(t_0)$, we obtain the evolution equations for $n_a, \alpha_a$ and $\Xi_a$. Integrating (2.21) by parts and varying with respect to $n_a, \alpha_a, \Xi_a$ with the boundary conditions

$$n_a(t = +\infty) = 1 \ , \ \ \alpha_a(t = +\infty) = 0 \ , \ \ \Xi_a^{-1}(t = +\infty) = 0 \ , \tag{3.2}$$

we obtain the evolution equations for $n_d, \alpha_d$ and $\Xi_d$. In general the evolution equations for $n_d, \alpha_d, \Xi_d$ and those for $n_a, \alpha_a, \Xi_a$ are coupled. The solutions $n_d, \alpha_d, \Xi_d$ and $n_a, \alpha_a, \Xi_a$ depend on the sources.

By combining the evolution equations for $n_d, \alpha_d, \Xi_d$ with those for $n_a, \alpha_a, \Xi_a$, the dynamical equations for the expectation values $\alpha_b, \Xi_b$ and $\alpha_c, \Xi_c$ can then be written in the following compact form :

$$i\,\dot{\alpha}_b = \tau\, w^b \ , \tag{3.3}$$

$$i\,\dot{\alpha}_c = \tau\ \left(w^c - w^c_K\right)\ , \tag{3.4}$$

$$i\,\dot{\Xi}_b = 2\left(\Xi_b\, \mathcal{H}^b\, \tau - \tau\, \mathcal{H}^b\, \Xi_b\right)\ , \tag{3.5}$$

$$i\,\dot{\Xi}_c = 2\ \left(\Xi_c\ \left(\mathcal{H}^c - \mathcal{I}^c_K\right)\ \tau - \tau\ \left(\mathcal{H}^c - \mathcal{I}^c_K\right)\ \Xi_c\right)\ , \tag{3.6}$$

where $\tau$ is the $2 \times 2$ matrix

$$\tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ . \tag{3.7}$$

The vector $w^b$ and the matrix $\mathcal{H}^b$ are defined from the variation of $< H >_b = Tr(\mathcal{T}_b H)$ :

$$\delta < H >_b = \int d^dx\ \tilde{w}^b(\vec{x}, t)\ \delta\beta_b(\vec{x}, t) - \frac{1}{2} \int d^dx\ d^dy\ tr\mathcal{H}^b(\vec{x}, \vec{y}, t)\ \delta\Xi_b(\vec{x}, \vec{y}, t) \ . \tag{3.8}$$

Similarly, $w^c, \mathcal{H}^c$ and $w^c_K, \mathcal{I}^c_K$ are defined from the variation of $< H >_c = Tr(\mathcal{T}_c H)$ and from the variation of the source term $K_c$.

Therefore, using a convenient parametrization for the two variational objects $\mathcal{D}(t)$ and $\mathcal{A}(t)$, we have been able to obtain dynamical equations in a very compact form even for finite values of the sources. The dynamical equations for $\alpha_d, \Xi_d$ and $\alpha_a, \Xi_a$ have a more complicated form. They are given in appendix B, where we write also the explicit expressions of the vector $w$ and the matrices $\mathcal{H}$ and $\mathcal{I}^c_K$ in the case of the $\lambda\Phi^4$ theory. In spite of their form, the equations (3.3)-(3.6) are coupled because the solutions $\alpha_b, \Xi_b$ and $\alpha_c, \Xi_c$ do not satisfy simple boundary conditions. The expansion in powers of the sources of the stationarity conditions (3.3) and (3.6) will provide approximate dynamical equations for the expectations and correlations functions defined in eqs. (2.5)-(2.9).

# 4 Expansion in powers of the sources

We will use upper index $^{(0)}$ for the solutions of the dynamical equations when there are no sources. The limit with vanishing source corresponds to $\alpha_a^{(0)} = 0$ and $\Xi_a^{-1^{(0)}} = 0$, $\alpha_b^{(0)} = \alpha_c^{(0)} = \alpha_d^{(0)}$ and $\Xi_b^{(0)} = \Xi_c^{(0)} = \Xi_d^{(0)}$; we have also $w^{b(0)} = w^{c(0)} = w^{(0)}$ and $\mathcal{H}^{b(0)} = \mathcal{H}^{c(0)} = \mathcal{H}^{(0)}$. In this limit, the dynamical equations (B.1) and (B.2) for $\alpha_d$ et $\Xi_d$ become (we supress the index $d$) :

$$i\,\dot{\alpha}^{(0)} = \tau\, w^{(0)} \ , \tag{4.1}$$

$$i\,\dot{\Xi}^{(0)} = -\left[\left(\Xi^{(0)} + \tau\right)\, \mathcal{H}^{(0)}\, \left(\Xi^{(0)} - \tau\right) - \left(\Xi^{(0)} - \tau\right)\, \mathcal{H}^{(0)}\, \left(\Xi^{(0)} + \tau\right)\right] \ . \tag{4.2}$$

These equations are the analog for the $\lambda\Phi^4$ theory of the time-dependent Hartree-Bogoliubov (TDHB) equations in non-relativistic physics. They are equivalent to the dynamical equations obtained in reference [6] where the authors used an alternative form of the Balian-Vénéroni variational principle suited to the evaluation of single-time expectation values [7].

The first derivatives of $W(J, t_0) = -i\, \ln \mathcal{Z}_{st}$ with respect to the sources are equal to the expectations values with the index $c$. Indeed, the functional $\mathcal{Z}$ depend on the sources

both explicitly and implicitly since the approximate solutions for $\mathcal{D}(t)$ and $\mathcal{A}(t)$ depend on the sources. However at the stationary point, only the explicit dependence contributes to the first derivative :

$$\frac{\delta \mathcal{Z}}{\delta J_j(t)} = i\, Tr\left(\mathcal{D}(t)\mathcal{A}(t)Q_j\right) \ , \tag{4.3}$$

which gives for instance : $\frac{\delta W}{\delta J_\Phi(\vec{x},t)} = \varphi_c(\vec{x},t)$. The expressions for the second derivatives of $W$ are much more complicated. The introduction of sources coupled to the composite operators $\Phi(\vec{x})\Phi(\vec{y}), \Phi(\vec{x})\Pi(\vec{y})$ and $\Pi(\vec{x})\Pi(\vec{y})$ together with eq. (4.3), allows us to obtain dynamical equations for two-time correlation functions with three or four field operators merely from the expansion of the expectation values $\alpha_c$ et $\Xi_c$ at the first order in powers of the sources.

From the first order corrections $\alpha_c - \alpha^{(0)}$ and $\Xi_c - \Xi^{(0)}$, we define the two-time correlation functions $\beta$ and $\Sigma$ ($\beta$ is a vector and $\Sigma$ is a matrix) :

$$\begin{aligned} \alpha_c(\vec{x},t) - \ & \alpha^{(0)}(\vec{x},t) \simeq i \int_{t_0}^{\infty} dt'' \left\{ \int d^d y\, \beta^\Phi(\vec{x},\vec{y},t,t'')\, J^\Phi(\vec{y},t'') + \beta^\Pi\, J^\Pi \right. \\ & \left. + \int d^d x_1 d^d x_2\, \beta^{\Phi\Phi}(\vec{x},\vec{x}_1,\vec{x}_2,t,t'')\, J^{\Phi\Phi}(\vec{x}_1,\vec{x}_2,t'') + \beta^{\Phi\Pi}\, J^{\Phi\Pi} + \beta^{\Pi\Pi}\, J^{\Pi\Pi} \right\} \ , \end{aligned} \tag{4.4}$$

$$\begin{aligned} \Xi_c(\vec{x},\vec{y},t) - \ & \Xi^{(0)}(\vec{x},\vec{y},t) \simeq i \int_{t_0}^{+\infty} dt'' \left\{ \int d^d x_1\, \Sigma^\Phi(\vec{x},\vec{y},,\vec{x}_1,t,t'')\, J^\Phi(\vec{x}_1,t'') + \Sigma^\Pi\, J^\Pi \right. \\ & \left. + \int d^d x_1 d^d x_2\, \Sigma^{\Phi\Phi}(\vec{x},\vec{y},\vec{x}_1,\vec{x}_2,t,t'')\, J^{\Phi\Phi}(\vec{x}_1,\vec{x}_2,t'') + \Sigma^{\Phi\Pi}\, J^{\Phi\Pi} + \Sigma^{\Pi\Pi}\, J^{\Pi\Pi} \right\} \ . \end{aligned} \tag{4.5}$$

We have

$$\beta_1^{\Phi\Phi}(\vec{x},\vec{y},\vec{z},t,t") = \Sigma_{11}^\Phi(\vec{x},\vec{y},\vec{z},t,t") \ . \tag{4.6}$$

The functions $\beta$ and $\Sigma$ provide approximations for the exact two-time correlation functions $C^2$, $C^3$ and $C^4$ defined by eqs. (2.7)-(2.9):

$$C^2_{\Phi\Phi}(\vec{x},\vec{y},t,t") \simeq \beta_1^\Phi(\vec{x},\vec{y},t,t") \ , \tag{4.7}$$

$$\begin{aligned} C^3(\vec{x},\vec{y},\vec{z},t,t") \simeq \ & \beta_1^{\Phi\Phi}(\vec{x},\vec{y},\vec{z},t,t") \\ & -\varphi^{(0)}(\vec{y},t")\, \left(\beta_1^\Phi(\vec{x},\vec{z},t,t") + \varphi^{(0)}(\vec{x},t)\varphi^{(0)}(\vec{z},t")\right) \ , \\ & -\varphi^{(0)}(\vec{z},t")\, \left(\beta_1^\Phi(\vec{x},\vec{y},t,t") + \varphi^{(0)}(\vec{x},t)\varphi^{(0)}(\vec{y},t")\right) \end{aligned} \tag{4.8}$$

$$
\begin{aligned}
C^4(\vec{x},\vec{y},\vec{z},\vec{u},t,t") \simeq\ & \Sigma^{\Phi\Phi}_{11}(\vec{x},\vec{y},\vec{z},\vec{u},t,t") \\
& -\Big(b^{\Phi}_1(\vec{x},\vec{z},t,t')+\varphi^{(0)}(\vec{x},t)\,\varphi^{(0)}(\vec{z},t")\Big)\ \Big(b^{\Phi}_1(\vec{y},\vec{u},t,t')+\varphi^{(0)}(\vec{y},t)\,\varphi^{(0)}(\vec{u},t")\Big) \\
& -\Big(b^{\Phi}_1(\vec{x},\vec{u},t,t')+\varphi^{(0)}(\vec{x},t)\,\varphi^{(0)}(\vec{u},t")\Big)\ \Big(b^{\Phi}_1(\vec{y},\vec{z},t,t')+\varphi^{(0)}(\vec{y},t)\,\varphi^{(0)}(\vec{z},t")\Big)
\end{aligned}
\quad . \tag{4.9}
$$

The expansion in powers of the sources of the dynamical equations (3.4) and (3.6) for $\beta_c$ et $\Xi_c$ yields :

$$
i\,\delta\dot{\beta}_c = \tau\ (\delta w^c - \delta w^c_K)\ , \tag{4.10}
$$

$$
\begin{aligned}
i\,\delta\dot{\Xi}_c = 2\quad & \Big[\delta\Xi_c\,\mathcal{H}^{(0)}\,\tau - \tau\,\mathcal{H}^{(0)}\,\delta\Xi_c \\
& +\Xi^{(0)}\ (\delta\mathcal{H}^c - \delta\mathcal{I}^c_K)\ \tau - \tau\ (\delta\mathcal{H}^c - \delta\mathcal{I}^c_K)\ \Xi^{(0)}\Big]
\end{aligned}
\quad . \tag{4.11}
$$

In these equations the matrix $\mathcal{H}$ has to be evaluated for the TDHB solutions $\alpha^{(0)},\Xi^{(0)}$ of eqs. (4.1)-(4.2). The variations of $w^c_K(\vec{x},t)$ (eqs. (B.14)-(B.15)) and $\mathcal{I}^c_K(\vec{x},\vec{y},t)$ (eqs. (B.16)-(B.18)) with respect to $J(t")$ will give terms proportional to $\delta(t-t")$. From the variations $\delta w^c$ and $\delta\mathcal{H}^c$, we define the matrices $t_{ij}, T_{i,jk}, r_{ij,k}, R_{ij,kl}$ (which are the analogs of the RPA kernel of Balian and Vénéroni [1]) :

$$
\delta w^c_i(\vec{x},t) = t_{ij}(\vec{x},t)\,\delta\alpha^c_j(\vec{x},t) - \frac{1}{2}\int d^dy\, T_{i,jk}(\vec{x},\vec{y},t)\,\delta\Xi^c_{kj}(\vec{y},\vec{x},t)\ , \tag{4.12}
$$

$$
\delta\mathcal{H}^c_{ij}(\vec{x},\vec{y},t) = r_{ij,k}(\vec{x},\vec{y},t)\,\delta\alpha^c_k(\vec{y},t) - \frac{1}{2}\int d^dz\, R_{ij,kl}(\vec{x},\vec{z},t)\,\delta\Xi^c_{lk}(\vec{z},\vec{y},t)\ . \tag{4.13}
$$

These matrices depend on the TDHB solutions $\alpha^{(0)}$ and $\Xi^{(0)}$. Their expressions in the case of the $\lambda\Phi^4$ theory are given in appendix B. With these notations, the dynamical equations for the two-time and three-point correlation functions $\beta^{\Phi}$ and $\Sigma^{\Phi}$ write :

$$
\begin{aligned}
i\,\tfrac{d}{dt}\beta^{\Phi}(\vec{x},\vec{y},t,t") = \tau\quad & \Big[t(\vec{x},t)\,\beta^{\Phi}(\vec{x},\vec{y},t,t") - \frac{1}{2}\,T_{,(jk}(\vec{x},\vec{x},t)\,\Sigma^{\Phi}_{kj)}(\vec{x},\vec{x},\vec{y},t,t") \\
& +i\begin{pmatrix}1\\0\end{pmatrix}\delta(\vec{x}-\vec{y})\,\delta(t-t")\Big]
\end{aligned}
\quad , \tag{4.14}
$$

$$
\begin{aligned}
i\,\tfrac{d}{dt}\Sigma^{\Phi}_{ij}(\vec{x},\vec{y},\vec{x}_1,t,t") =\ & 2\,\Big[\Sigma^{\Phi}\,\mathcal{H}^{(0)}\,\tau - \tau\,\mathcal{H}^{(0)}\,\Sigma^{\Phi} \\
& +\Xi^{(0)}\,r_{,(k}\,\beta^{\Phi}_{k)}\,\tau - \tau\,r_{,(k}\,\beta^{\Phi}_{k)}\,\Xi^{(0)} \\
& -\frac{1}{2}\,\Xi^{(0)}\,R_{,(kl}\,\Sigma^{\Phi}_{lk)}\,\tau + \frac{1}{2}\,\tau\,R_{,(kl}\,\Sigma^{\Phi}_{lk)}\Xi^{(0)}\Big]_{ij}
\end{aligned}
\quad . \tag{4.15}
$$

We obtain similar equations for the three-point functions $\beta^{\Phi\Phi},\beta^{\Phi\Pi}$, $\beta^{\Pi\Pi}$ and the four-point functions $\Sigma^{\Phi\Phi},\Sigma^{\Phi\Pi}$ et $\Sigma^{\Pi\Pi}$. These dynamical equations are not sufficient since

the boundary conditions are $\alpha_a(t=+\infty)=0, \Xi_a^{-1}(t=+\infty)=0$ and $\alpha_d(\vec{x},t_0)=\alpha_0(\vec{x})$, $\Xi_d(\vec{x},\vec{y},t_0)=\Xi_0(\vec{x},\vec{y})$. We will use the dynamical equations for the functions $l$ defined from the expansion of $\Xi_a^{-1}$ at the first order :

$$\begin{aligned}\Xi_a^{-1}(\vec{x},\vec{y},t)\simeq\ & i\int_{t_0}^{\infty}dt''\ \Big\{\int d^dx_1\, l^{\Phi}(\vec{x},\vec{y},\vec{x}_1,t'',t)\ J^{\Phi}(\vec{x}_1,t'')+l^{\Pi}\ J^{\Pi}\\ &+\int d^dx_1d^dx_2\, l^{\Phi\Phi}(\vec{x},\vec{y},\vec{x}_1,\vec{x}_2,t'',t)\ J^{\Phi\Phi}(\vec{x}_1,\vec{x}_2,t'')+l^{\Phi\Pi}\ J^{\Phi\Pi}+l^{\Pi\Pi}\ J^{\Pi\Pi}\Big\}\ .\end{aligned} \tag{4.16}$$

We have also at the first order :

$$\alpha_c-\alpha_d\simeq\left(\tau\,\Xi_a^{-1}-\Xi_a^{-1}\,\Xi^{(0)}\right)\ \alpha^{(0)} \tag{4.17}$$

and

$$\Xi_c-\Xi_d\simeq-\left(\Xi_d-\tau\right)\ \left(\Xi^{(0)}\,\Xi_a^{-1}+\Xi_a^{-1}\,\tau\right)\ . \tag{4.18}$$

At the initial time $t_0$,

$$\alpha_c(t_0)-\alpha_0\simeq\left(\tau\,\Xi_a^{-1}-\Xi_a^{-1}\,\Xi_0\right)\ \alpha_0\ , \tag{4.19}$$

and

$$\Xi_c(t_0)-\Xi_0\simeq-\left(\Xi_0-\tau\right)\ \left(\Xi_0\,\Xi_a^{-1}+\Xi_a^{-1}\,\tau\right)\ . \tag{4.20}$$

This yields the following relations between the functions $\beta^{\Phi}(t_0,t'')$, $\Sigma^{\Phi}(t_0,t'')$ and the function $l^{\Phi}(t'',t_0)$ :

$$\beta^{\Phi}(\vec{x},\vec{x}_1,t_0,t'')=\int d^dz_1d^dz_2\left(\tau\, l^{\Phi}(\vec{x},\vec{z}_1,\vec{x}_1,t'',t_0)\,\delta(\vec{z}_1-\vec{z}_2)-l^{\Phi}(\vec{x},\vec{z}_1,\vec{x}_1,t'',t_0)\,\Xi_0(\vec{z}_1,\vec{z}_2)\right)\alpha_0(\vec{z}_2)\ , \tag{4.21}$$

$$\begin{aligned}\Sigma^{\Phi}(\vec{x},\vec{y},\vec{x}_1,t_0,t'')=\ &-\int d^dz_1\ d^dz_2\ \left(\Xi_0(\vec{x},\vec{z}_1)-\tau\right)\\ &\times\left(\Xi_0(\vec{z}_1,\vec{z}_2)\,l^{\Phi}(\vec{z}_2,\vec{y},\vec{x}_1,t'',t_0)+\delta(\vec{z}_2-\vec{y})\,l^{\Phi}(\vec{z}_1,\vec{y},\vec{x}_1,t'',t_0)\,\tau\right)\end{aligned}\ . \tag{4.22}$$

We have similar relations between the functions $\beta^{\Phi\Phi}, \Sigma^{\Phi\Phi}$ and $l^{\Phi\Phi}$.

To lowest order, the dynamical equation for $\Xi_a^{-1}$ reads :

$$i\,\frac{d}{dt}\left(\Xi_a^{-1}\right)=\mathcal{H}^c-\mathcal{H}^b-\mathcal{I}_K^c-2\,\Xi_a^{-1}\tau\mathcal{H}^{(0)}+2\,\mathcal{H}^{(0)}\tau\Xi_a^{-1}\ . \tag{4.23}$$

From the expansion (4.16) for $\Xi_a^{-1}$ and the variations $\delta\mathcal{H}^c-\delta\mathcal{H}^b$, we obtain :

$$\begin{aligned} i\,\tfrac{d}{dt}l(\vec{x},\vec{y},\vec{x}_1,t'',t)_{ij}=\ &-2\,r_{ij,k}\,\tau_{kl}\,l_{lm}\,\alpha_m^{(0)}-R_{ij,kl}\,\left(\Xi^{(0)}\,l\,\tau-\tau\,\Xi^{(0)}\,l\right)_{lk}\\ &+\left[-2\,l\tau\mathcal{H}^{(0)}+2\,\mathcal{H}^{(0)}\,\tau\,l+i\,\delta\mathcal{I}_K^c\,\delta(t-t'')\right]_{ij}\end{aligned}\ . \tag{4.24}$$

We have $l(\vec{x},\vec{y},t",t)=0$ for $t>t"$ and :

$$l^{\Phi\Phi}(\vec{x},\vec{y},\vec{x}_1,\vec{x}_2,t",t")=1\ ,\ \ l^{\Phi\Pi}(\vec{x},\vec{y},\vec{x}_1,\vec{x}_2,t",t")=2i\ ,\ \ l^{\Pi\Pi}(\vec{x},\vec{y},\vec{x}_1,\vec{x}_2,t",t")=-1\ , \tag{4.25}$$

$$l^{\Phi}(\vec{x},\vec{y},\vec{x}_1,t",t")=l^{\Pi}(\vec{x},\vec{y},\vec{x}_1,t",t")=0\ . \tag{4.26}$$

Therefore, to obtain approximations for the two-time causal correlation functions $C^2, C^3$ and $C^4$, we need first to solve the TDHB equations (4.1) and (4.2) for $\alpha^{(0)}$ and $\Xi^{(0)}$ and to evaluate the matrices $r, R, t$ and $T$. Then we solve the equation (4.24) for $l^{\Phi}(\vec{x},\vec{y},\vec{x}_1,t",t)$ and $l^{\Phi\Phi}(\vec{x},\vec{y},\vec{x}_1,\vec{x}_2,t",t)$ $(t" > t' \geq t_0)$ backward from $t"$ to $t_0$ with the boundary conditions (4.25)-(4.26). From the relations (4.21) and (4.22), we obtain $\beta^{\Phi}(\vec{x},\vec{x}_1,t_0,t"), \beta^{\Phi\Phi}(\vec{x},\vec{x}_1,\vec{x}_2,t_0,t")$ , $\Sigma^{\Phi}(\vec{x},\vec{y},\vec{x}_1,t_0,t")$ and $\Sigma^{\Phi\Phi}(\vec{x},\vec{y},\vec{x}_1,\vec{x}_2,t_0,t")$ and with these boundary conditions we solve the dynamical equations for $\beta(t,t")$ and $\Sigma(t,t")$ forward from $t$ to $t"$ .

In the TDHB approximation, the three-point function $C^3$ and the four-point function $C^4$ vanish. Using the Balian-Vénéroni variational approach, we are able to obtain approximations for $C^3$ and $C^4$ given by eqs. (4.8) and (4.9) which differ from this naïve result.

In order to obtain approximations for the two-time anticausal functions, it is sufficient to replace in the functional (2.10)-(2.11) the term $K_c$ by the analog term $K_b$. We can therefore derive a consistent approximate formula for the response functions which involve the retarded commutator.

When the two times coïncide in the function $\beta_1^{\Phi}$, we will obtain an approximation for the function $G(\vec{x},\vec{x}_1,t)$ which will differ from the TDHB solution $G^{(0)}$ (they satisfy however the same initial condition $G(\vec{x},\vec{x}_1,t_0)=G^{(0)}(\vec{x},\vec{x}_1,t_0)=G_0(\vec{x},\vec{x}_1)$ ). We investigate the static situation in a subsequent paper where we use a form of the Balian-Vénéroni variational principle which optimizes both the initial state and the dynamics [1]. We will examine the divergences which appear in the variational equations and study how the approximations obtained for the functions $G, C^3$ and $C^4$ are related to a few physical quantities of the theory, i. e. the renormalized mass of the particle and the renormalized

coupling constant.

In conclusion, using an appropriate parametrization, we have shown how to evaluate variationally the generating functional for multi-time correlation functions in a quantum field theory out of equilibrium when the initial state is given and equal to a Gaussian state. We have obtained dynamical equations in a compact form whose expansion at the first order in power of the sources provides approximations for the two-time correlation functions.

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**Appendix A**

In this appendix we give the expressions of the expectation values $\alpha_b, \alpha_c, \Xi_b$ and $\Xi_c$ as functions of $\alpha_d, \alpha_a, \Xi_d$ and $\Xi_a$.

$$\alpha_b = (\Xi_a - \tau)\ \frac{1}{\Xi_a + \Xi_d}\ \alpha_d + (\Xi_d + \tau)\ \frac{1}{\Xi_a + \Xi_d}\ \alpha_a\ , \tag{A.1}$$

$$\alpha_c = (\Xi_d - \tau)\ \frac{1}{\Xi_a + \Xi_d}\ \alpha_a + (\Xi_a + \tau)\ \frac{1}{\Xi_a + \Xi_d}\ \alpha_d\ , \tag{A.2}$$

where $\tau$ is the $2 \times 2$ matrix :

$$\tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\ . \tag{A.3}$$

For the matrices $\Xi$, we have the following relations :

$$\Xi_b - \tau = (\Xi_a - \tau)\ \frac{1}{\Xi_a + \Xi_d}\ (\Xi_d - \tau)\ \ , \tag{A.4}$$

$$\Xi_c - \tau = (\Xi_d - \tau)\ \frac{1}{\Xi_a + \Xi_d}\ (\Xi_a - \tau)\ \ . \tag{A.5}$$

**Appendix B**

In this appendix, we give the evolution equations for $\alpha_d, \Xi_d$ and $\alpha_a, \Xi_a$.

$$\begin{aligned} 2i\,\dot{\alpha}_d = \; & -\Big[(\Xi_d+\tau)\,(\mathcal{H}^c-\mathcal{I}_K^c)\,(\Xi_d-\tau)-(\Xi_d-\tau)\;\mathcal{H}^b\;(\Xi_d+\tau)\Big]\;\tau\,\alpha_{b-c} \\ & +\Xi_d\left(w^c-w^b\right)+\tau\;\left(w^c+w^b\right)-(\Xi_d+\tau)\;w_K^c \end{aligned} \tag{B.1}$$

$$i\,\dot{\Xi}_d = -\Big[(\Xi_d+\tau)\,(\mathcal{H}^c-\mathcal{I}_K^c)\,(\Xi_d-\tau)-(\Xi_d-\tau)\;\mathcal{H}^b\;(\Xi_d+\tau)\Big] \tag{B.2}$$

$$\begin{aligned} 2i\,\dot{\alpha}_a = \; & \Big[(\Xi_a-\tau)\,(\mathcal{H}^c-\mathcal{I}_K^c)\,(\Xi_a+\tau)-(\Xi_a+\tau)\;\mathcal{H}^b\;(\Xi_a-\tau)\Big]\;\tau\,\alpha_{c-b} \\ & +\Xi_a\left(w^b-w^c\right)+\tau\;\left(w^c+w^b\right)-(\Xi_a-\tau)\;w_K^c \end{aligned} \quad , \tag{B.3}$$

$$i\,\dot{\Xi}_a = -\Big[(\Xi_a-\tau)\,(\mathcal{H}^c-\mathcal{I}_K^c)\,(\Xi_a+\tau)-(\Xi_a+\tau)\;\mathcal{H}^b\;(\Xi_a-\tau)\Big] \tag{B.4}$$

The expressions for the vector $w$ and the matrices $\mathcal{H}$ and $\mathcal{I}$ are the following :

$$\tilde{w}^b(\vec{x},t)_1 = \frac{\delta < H >_b}{\delta\varphi_b(\vec{x},t)} = -f^b(\vec{x},t) \quad , \quad \tilde{w}^b(\vec{x},t)_2 = i\,\frac{\delta < H >_b}{\delta\pi_b(\vec{x},t)} = i\pi^b(\vec{x},t)\ , \tag{B.5}$$

$$\mathcal{H}^b(\vec{x},\vec{y},t)_{ij} = -2\frac{\delta < H >_b}{\delta\Xi_b(\vec{x},\vec{y},t)_{ji}}\ , \tag{B.6}$$

$$\mathcal{H}^b(\vec{x},\vec{y},t)_{11} = -\frac{\delta < H >_b}{\delta G_b(\vec{y},\vec{x},t)} = \frac{1}{2}\,g^b(\vec{x},\vec{y},t)\ , \tag{B.7}$$

$$\mathcal{H}^b(\vec{x},\vec{y},t)_{22} = +\frac{\delta < H >_b}{\delta S_b(\vec{y},\vec{x},t)} = \frac{1}{2}\,\delta(\vec{x}-\vec{y})\ , \tag{B.8}$$

$$\mathcal{H}^b(\vec{x},\vec{y},t)_{12} = 2i\,\frac{\delta < H >_b}{\delta T_b(\vec{y},\vec{x},t)} = 0\ . \tag{B.9}$$

For a self-interacting scalar field, we have :

$$f_b(\vec{x},t) = -\left(-\Delta+m_0^2+\frac{b}{6}\,\varphi_b^2(\vec{x},t)+\frac{b}{2}\,G_b(\vec{x},\vec{x},t)\right)\;\varphi_b(\vec{x},t)\ , \tag{B.10}$$

$$g_b(\vec{x},\vec{y},t) = -\left(-\Delta+m_0^2+\frac{b}{2}\,\varphi_b^2(\vec{x},t)+\frac{b}{2}\,G_b(\vec{x},\vec{x},t)\right)\;\delta(\vec{x}-\vec{y})\ . \tag{B.11}$$

From the variations of the source term $K_c$, we obtain :

$$\tilde{w}_K^c(\vec{x},t)_1 = \frac{\delta K_c}{\delta\varphi_c(\vec{x},t)} \quad , \quad \tilde{w}_K^c(\vec{x},t)_2 = i\,\frac{\delta K_c}{\delta\pi_c(\vec{x},t)}\ , \tag{B.12}$$

$$\mathcal{I}_K^c(\vec{x},\vec{y},t)_{ij} = -2\frac{\delta K_c}{\delta \Xi_c(\vec{y},\vec{x},t)_{ji}} \ . \tag{B.13}$$

$$w_K^c(\vec{x},t)_1 = J^{\Phi}(\vec{x},t) + 2\int d^dx_2 \, \Big( J^{\Phi\Phi}(\vec{x},\vec{x}_2,t)\,\varphi_c(\vec{x}_2,t) + J^{\Phi\Pi}(\vec{x},\vec{x}_2,t)\,\pi_c(\vec{x}_2,t)\Big) \ , \tag{B.14}$$

$$w_K^c(\vec{x},t)_2 = i\, J^{\Pi}(\vec{x},t) + 2\, i \int d^dx_2 \, \Big( J^{\Phi\Pi}(\vec{x},\vec{x}_2,t)\,\varphi_c(\vec{x}_2,t) + J^{\Pi\Pi}(\vec{x},\vec{x}_2,t)\,\pi_c(\vec{x}_2,t)\Big) \ , \tag{B.15}$$

$$\mathcal{I}_K^c(\vec{x},\vec{y},t)_{11} = -J^{\Phi\Phi}(\vec{x},\vec{y},t) \ , \tag{B.16}$$

$$\mathcal{I}_K^c(\vec{x},\vec{y},t)_{12} = -2i\, J^{\Phi\Pi}(\vec{x},\vec{y},t) \ , \tag{B.17}$$

$$\mathcal{I}_K^c(\vec{x},\vec{y},t)_{22} = J^{\Pi\Pi}(\vec{x},\vec{y},t) \ . \tag{B.18}$$

The expressions for the matrices $t, T, r$ and $R$ which appear in the dynamical equations for the two-time correlation functions are the following :

$$t_{11}(\vec{x},t) = -g^{(0)}(\vec{x},\vec{x},t) \ \ , \ \ t_{22} = -1 \ , \tag{B.19}$$

$$T_{1,11}(\vec{x},\vec{y},t) = \frac{b}{2}\,\varphi^{(0)}(\vec{x},t)\delta(\vec{x}-\vec{y}) \ \ , \ \ T_{2,jk} = 0 \ , \tag{B.20}$$

$$r_{11,1}(\vec{x},\vec{y},t) = -\frac{b}{2}\,\varphi^{(0)}(\vec{x},t)\delta(\vec{x}-\vec{y}) \ , \tag{B.21}$$

$$R_{11,11}(\vec{x},\vec{y},t) = \frac{b}{4}\,\delta(\vec{x}-\vec{y}) \ . \tag{B.22}$$

The other matrix elements are equal to zero.

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